Physical Interpretation of the Green's function.

Definition of delta function : $\delta(x, x_0)$.

Heat Conduction: (1-D)

$$u_t = K u_{xx} + \underbrace{Q(x)}_{\text{Sources of energy}} \quad , \qquad 0 < x < L$$

If steady $u_t = 0$ and $K \equiv 1$.

$$\Rightarrow \boxed{-u'' = Q(x)}$$

If BC's. $u(0) = 0$, $u(L) = 0$.

$Q(x)$: Represent a continuous distibution of sources along $0 < x < L$. At a given point x_0, the value $Q(x_0)$ represent the strength of the source located at x_0.

Ideal Case: i) Source is located at one point $x = x_0$ only.

ii) This source has unit strength.

The following notation will be used: $Q(x) = \delta(x, x_0)$.

i) $\delta(x, x_0) = 0, \quad x \neq x_0$

ii) $\displaystyle\int_0^L \delta(x, x_0)\,dx = 1$

Not a conventional function, $h(x)$. Any conv. function zero everywhere except at one point $x_0 \in (a, b)$,

$$\int_a^b h(x)\,dx = 0.$$

Obtaining Green's function using the delta function: $\delta(x,x_0)$

Let's consider the $\overbrace{\text{steady state heat conduction operator}}$ again: $-u''(x)$. → Same as 1-D Laplace.

And we will try to obtain the solution for a source of unit strength located at $x = x_0$.

$$\begin{cases} -u'' = \delta(x,x_0) & (6.1) \\ u(0)=0, \quad u(L)=0. & (6.2) \end{cases}$$

We will obtain this directly.

The first thing to verify is that the associated S-L EVP does not have $\lambda = 0$ as an eigenvalue(*). In our case, we already know this. We expect that under this condition BVP (6.1) and (6.2) has a unique solution.

We start the procedure to obtain a solution
Solving two BVP governed by the corresponding homogeneous equation.

Choose x_0 s.t. $\underset{0}{\vdash}\!\!\text{———}\!\!\underset{x_0}{+}\!\!\text{———}\!\!\underset{L}{\dashv}$

a) $\begin{cases} \boxed{-u''=0, \quad u(0)=0, \quad a<x<x_0} \\ \Rightarrow u(x) = A_1 x, \quad u(x_0^-) \text{ specified later} \end{cases}$ $\boxed{u^-(x) = A_1 x + B_1}$ $0 = u(0) = B_1$

b) $\begin{cases} -u''=0, \quad u(L)=0, \quad u(x_0^+) \text{ is specified later}, \quad x_0<x<b \\ \boxed{u^+(x) = -\frac{B_2}{L}x + B_2 = B_2\left(1-\frac{x}{L}\right).} \end{cases}$ $u_2(x) = A_2 x + B_2$, $0 = u_2(L) = A_2 L + B_2$, $\Rightarrow A_2 = -\frac{B_2}{L}$

(*) This can be shown directly by showing that the only soln. of $-u''=0$, $u(0)=0$, $u(L)=0$ is the trivial soln.

(I) At $x = x_0$, we impose the condition that the temperature be continuous

$$u^+(x_0^+) = u^-(x_0^-)$$

This implies

$$A_1 x_0 = B_2\left(1 - \frac{x_0}{L}\right) \Rightarrow \boxed{A_1 x_0 - \left(1 - \frac{x_0}{L}\right)B_2 = 0} \tag{7.1}$$

(II) A second condition (Jump condition) is obtained by formally integrating the equation: between $x_0 - \varepsilon$ and $x_0 + \varepsilon$.

$$-\int_{x_0-\varepsilon}^{x_0+\varepsilon} u''(x)\,dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x, x_0)\,dx = 1$$

$$\xRightarrow{\varepsilon \to 0} \quad \boxed{(u^+)'(x_0^+) - (u^-)'(x_0^-) = -1} \quad \text{Jump in the derivative.}$$

or

$$\boxed{-\frac{B_2}{L} - A_1 = -1} \Rightarrow \boxed{A_1 = 1 - \frac{B_2}{L}}$$

Substitution into (7.1)

$$\left(1 - \frac{B_2}{L}\right)x_0 - \left(1 - \frac{x_0}{L}\right)B_2 = 0 \Rightarrow \boxed{B_2 = x_0} \Rightarrow \boxed{A_1 = 1 - \frac{x_0}{L}}.$$

Therefore, the solution for the BVP (6.1)-(6.2) is given by

$$\boxed{u(x, x_0) = \begin{cases} \left(1 - \frac{x_0}{L}\right)x, & x < x_0 \\ \left(1 - \frac{x}{L}\right)x_0, & x > x_0 \end{cases} \qquad (7.2)}$$

$u(x, x_0)$ is Symmetric. In fact,

$$u(x_0, x) = \begin{cases} \left(1 - \frac{x}{L}\right)x_0, & x_0 < x \\ \left(1 - \frac{x_0}{L}\right)x, & x_0 > x \end{cases} = u(x, x_0)$$

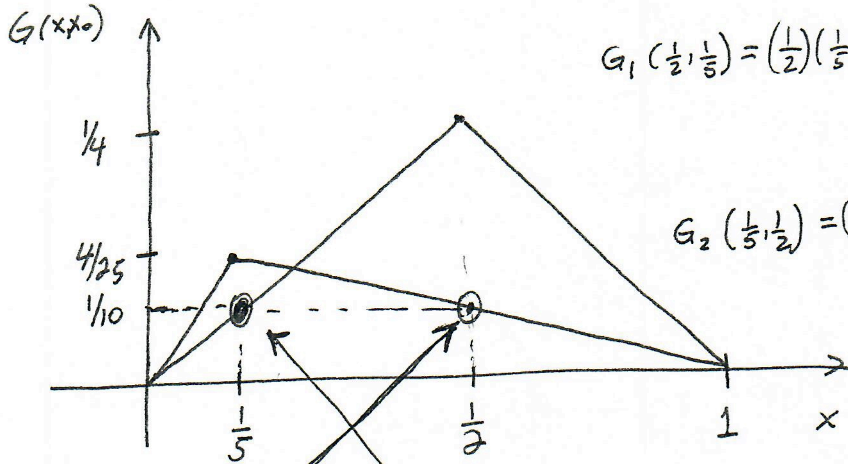
Green's functions given by formula (7.2) are clearly symmetric. This symmetry is also called Maxwell reciprocity.

$$\boxed{G(x_1, x_2) = G(x_2, x_1)}.$$

The graphs of Green's functions (7.2) $L=1$ for sources located at $x_{0_1} = \frac{1}{5}$, and $x_{0_2} = \frac{1}{2}$ are as follows

$$G_1(x,\tfrac{1}{5}) = \begin{cases} (1-\frac{1}{5})x, & x \le 1/5 \\ (1-x)\frac{1}{5}, & x > 1/5 \end{cases}$$

$$G_2(x,\tfrac{1}{2}) = \begin{cases} (1-\frac{1}{2})x, & x \le 1/2 \\ (1-x)\frac{1}{2}, & x > 1/2 \end{cases}$$



$$G_1(\tfrac{1}{2},\tfrac{1}{5}) = (\tfrac{1}{2})(\tfrac{1}{5}) = \tfrac{1}{10}$$

$$G_2(\tfrac{1}{5},\tfrac{1}{2}) = (\tfrac{1}{2})(\tfrac{1}{5}) = \tfrac{1}{10}$$

Therefore,

$$G_1(\tfrac{1}{2}, \tfrac{1}{5}) = G_2(\tfrac{1}{5}, \tfrac{1}{2})$$

Green's formula

Consider two functions $u(x)$ and $v(x)$ continuously differentiable twice.

$$(uv')' = \cancel{u'v'} + uv''$$
$$(vu')' = \cancel{v'u'} + vu''$$

$$(uv')' - (vu')' = uv'' - vu''$$

or $$uv'' - vu'' = \left((uv') - (vu')\right)'$$

Integrating betw 0 and L

$$\int_0^L (uv'' - vu'')\,dx = \int_0^L \left((uv') - (vu')\right)' dx$$

or $$\boxed{\int_0^L (uv'' - vu'')\,dx = \left[(uv') - (vu')\right]\Big|_0^L} \quad (8.2)$$

$$= u(L)v'(L) - v(L)u'(L) - u(0)v'(0) + v(0)u'(0)$$

So $$\boxed{\int_0^L (uv'' - vu'')\,dx = u(L)v'(L) - v(L)u'(L) - u(0)v'(0) + v(0)u'(0)} \quad (8.3)$$

BVP with homogeneous bondary cond. and a forcing fn.

$$-u''(x) = f(x), \quad u(0)=0, \quad u(L)=0.$$

Green's fn: $\quad -G''(x,x_0) = \delta(x,x_0) \quad G(0,x_0)=0, \quad G(L,x_0)=0$

Green's formula:

$$\int_0^L (uv'' - vu'')\,dx = \left[uv' - vu'\right]\Big|_0^L \qquad (11.1)$$

Replacing $v = G$.

$$\int_0^L u(x)\,G''(x,x_0)\,dx - \int_0^L G(x,x_0)\,u''(x)\,dx =$$

$$= \left[u(x)\,G'(x,x_0) - G(x,x_0)\,u'(x)\right]\Big|_0^L$$

or $$\int_0^L u(x)\left(-\delta(x,x_0)\right)dx - \int_0^L G(x,x_0)\left(-f(x)\right)dx = 0$$

$$\Rightarrow \quad -u(x_0) = -\int_0^L G(x,x_0)\,f(x)\,dx$$

or $$u(x_0) = \int_0^L G(x,x_0)\,f(x)\,dx \overset{\text{Symm of } G(x,x_0)}{=} \int_0^L G(x_0,x)\,f(x)\,dx$$

interchanging roles $x \leftrightarrow x_0$

$$\boxed{u(x) = \int_0^L G(x,x_0)\,f(x_0)\,dx_0} \qquad (11.2)$$

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This is exactly the Green's function $G(x,x_0)$ obtained previously by variation of parameters.

Now, we can describe the Green's function $G(x,x_0)$ as the response of the system at a point x, due to a source of unit strength located at x_0.

Comparison of the two solutions for the two BVP:

(I) $\quad -u'' = f(x), \quad 0 < x < L, \quad u(0)=0, \; u(L)=0$

(II) $\quad -v'' = \delta(x-x_s), \quad 0 < x < L, \quad v(0)=0, \; v(L)=0$

For (I), we obtained

$$\boxed{u(x) = \int_0^L G(x,x_0) f(x_0)\,dx_0} \qquad (8.1)$$

For II, we obtained $\quad v(x) = \int_0^L G(x,x_0)\,\delta(x_0-x_s)\,dx_0 = G(x,x_s).$

$$\boxed{v(x) = G(x,x_s).}$$

Interpretation:

Soln. of (II) corresponds to the response of the system to a concentrated source of unit strength located at $x = x_s$.

Soln. of (I) corresponds to the response of the system to a continuous distribution of sources on the interval $[0,L]$, each one of strength $f(x_0)$ at $x = x_0$.

Bound. Value Problem with nonhomogeneous B.C's.

$$\boxed{-u''(x) = f(x), \qquad u(0) = \alpha, \quad u(L) = \beta.}$$

$$\boxed{-G''(x,x_0) = \delta(x,x_0), \qquad G(0,x_0) = 0, \quad G(L,x_0) = 0}$$

Subst. into (11.1) Green's formula:

$$\int_0^L u(x)\left(-\delta(x,x_0)\right)dx - \int_0^L G(x,x_0)\left(-f(x)\right)dx =$$

$$= \underbrace{u(L)}_{=\beta} G'(L,x_0) - \cancel{G(L,x_0)}^{\,0} u'(L) - \underbrace{u(0)}_{=\alpha} G'(0,x_0) + u'(0)\cancel{G(0,x_0)}^{\,0}$$

$$= \beta \frac{dG}{dx}(L,x_0) - \alpha \frac{dG}{dx}(0,x_0)$$

$$\Rightarrow \boxed{-u(x_0) + \int_0^L G(x,x_0) f(x)\,dx = \beta \frac{dG}{dx}(L,x_0) - \alpha \frac{dG}{dx}(0,x_0)} \quad (12.0)$$

Using the Symmetry of $G(x,x_0)$: $G(x,x_0) = G(x_0,x)$ (12.1)

$$\Rightarrow \frac{dG}{dx}(x,x_0) = \frac{dG}{dx}(x_0,x). \qquad (12.2)$$

Using (12.1) and (12.2) and reversing roles of $x_0 \leftrightarrow x$ in (12.0), turn

we obtain

$$u(x) = \int_0^L G(x,x_0) f(x_0)\,dx_0 - \beta \frac{dG}{dx_0}(x,L) + \alpha \frac{dG}{dx_0}(x,0)$$

or

$$\boxed{u(x) = \int_0^L G(x,x_0) f(x_0)\,dx_0 + \beta \frac{x}{L} + \alpha\left(1 - \frac{x}{L}\right)} \quad (12.2)$$

Summary

Green's function representation for BVP with nonhomogeneous BCs.

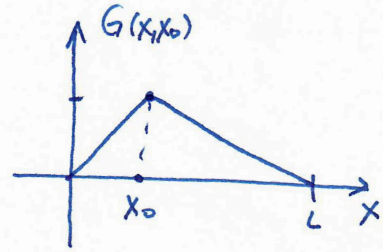
BVP:

$$\begin{cases} -u''(x) = f(x), & 0<x<L \\ u(0)=\alpha, \quad u(L)=\beta. \end{cases}$$

BVP Satisfied by associated Green's function:

$$\begin{cases} -G''(x,x_0) = \delta(x-x_0), & 0<x<L, \quad x_0 \in (0,L) \\ G(0,x_0)=0, \quad G(L,x_0)=0 \end{cases}$$

$$G(x,x_0) = \begin{cases} \left(1-\frac{x_0}{L}\right)x, & x<x_0 \\ \left(1-\frac{x}{L}\right)x_0, & x>x_0 \end{cases}$$



Green's 2nd identity or Simply Green's formula

$$\int_0^L [uv''-vu'']\,dx = [uv'-vu']\Big|_0^L$$

Replacing $v=G$ and using properties of G and δ.

$$\boxed{u(x) = \int_0^L G(x,x_0)\,f(x_0)\,dx_0 + \beta\frac{x}{L} - \alpha\left(1-\frac{x}{L}\right)}$$